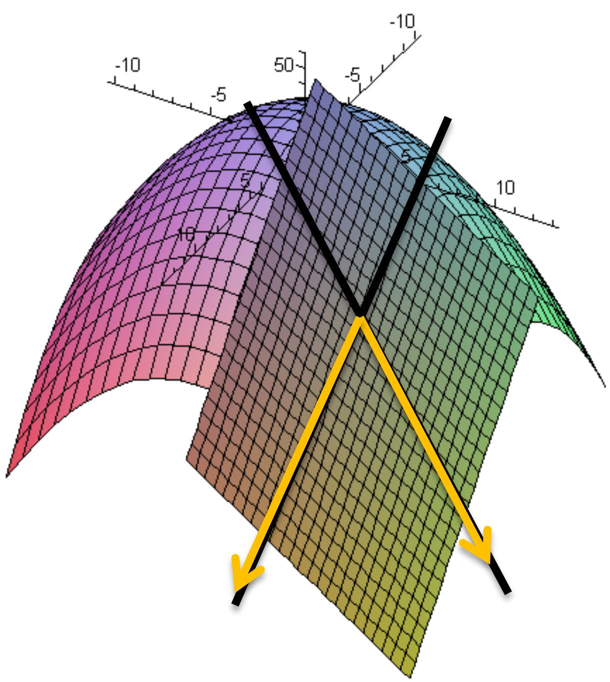
Closing Tue: 14.3 (part 2), 14.4
Closing Thu: 14.7

14.4 Tangent Planes
The <u>tangent plane</u> to a surface at a point is the plane that contains all tangent lines at that point.

Derivation Example:

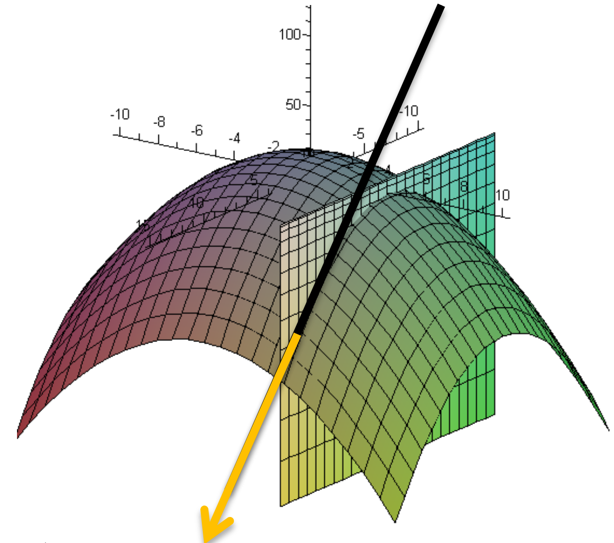
$$z = f(x,y) = 15 - x^2 - y^2 \text{ at } (7,4)$$

Step 1: Compute $z_0 = f(7,4) = -50$

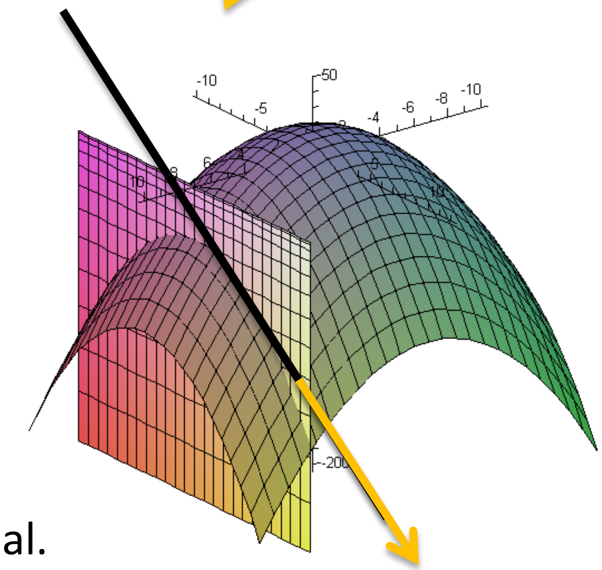


Step 2: Get partials. Form direction vectors.

$f_x(x,y) = -2x$
$f_x(7,4) = -14$
$<1,0,f_x(x_0,y_0)>$
$= <1,0,-14>$



$f_y(x,y) = -2y$
$f_y(7,4) = -8$
$<0,1,f_y(x_0,y_0)>$
$= <0,1,-8>$



Step 3: Get normal.
$< 1, 0, -14 > \times < 0, 1, -8 > = < 14, 8, 1 >$

Tangent Plane: $14(x-7) + 8(y-4) + (z+50) = 0$

$$z + 50 = -14(x-7) - 8(y-4)$$

In general, we see that we can find the tangent plane for $z = f(x,y)$ at (x_0, y_0) by:

1. Computing $z_0 = f(x_0, y_0)$.
2. $\langle 1,0, f_x(x_0, y_0)\rangle$ = tangent vector in x-dir.
   $\langle 0,1, f_y(x_0, y_0)\rangle$ = tangent vector in y-dir.
3. Normal:
   $$\vec{\boldsymbol{n}} = \langle 1,0, f_x(x_0, y_0)\rangle \times \langle 0,1, f_y(x_0, y_0)\rangle$$
   $$= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$$
   = normal to the surface at (x_0,y_0)

Tangent Plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which is typically written as

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example:

Find the tangent plane for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at $(x,y) = (2,1)$.

Some Applications of the Tangent Plane

1. *Linear Approximation*:
   "Near" the points (x_0,y_0) the z-values on the plane will be very close to the z-values on the surface. So we can use the tangent plane as an approximation of the z-values on the surface.

2. *Differentials*:
   Often this linear approximation idea is stated in terms of differences.
   Label the actual changes on the surface by
   $$\Delta x = x - x_0,\ \Delta y = y - y_0,\ \Delta z = f(x,y) - f(x_0,y_0)$$
   Label the changes on the tangent plane by:
   $$dx = x - x_0,\ dy = y - y_0,\ dz = z - z_0$$
   For small values of dx and dy, we get
   $$dz \approx \Delta z$$

Example:
Find the linear approximation and total differential for
$$f(x,y) = x^2 + 3y^2x - y^3$$
at $(x,y) = (2,1)$.

HW Hint:
Use differentials to estimate the amount of metal in a closed cylindrical can that is 18 cm high and 8 cm in diameter if the metal in the top and the bottom is 0.2 cm thick and the metal in the sides is 0.05 cm thick. (Round your answer to two decimal places.)

Notes: *Estimating change in volume*

$dV = ???$

Let r = radius and h = height,
then $V = \pi r^2 h$
Total differential (same as tangent plane):

$$dV = 2\pi r h \, dr + \pi r^2 \, dh$$

now plug in numbers:

$h = 18$,
$r = 4$,
$dh = 0.2 + 0.2 = 0.4$
$dr = 0.05$